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Statistical regularities in the return intervals of volatility

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Abstract. We discuss recent results concerning statistical regularities in the return intervals of volatility in financial markets. In particular, we show how the analysis of volatility return intervals, defined as the time between two volatilities larger than a given threshold, can help to get a better understanding of the behavior of financial time series. We find scaling in the distribution of return intervals for thresholds ranging over a factor of 25, from 0.6 to 15 standard deviations, and also for various time windows from one minute up to 390 min (an entire trading day). Moreover, these results are universal for different stocks, commodities, interest rates as well as currencies. We also analyze the memory in the return intervals which relates to the memory in the volatility and find two scaling regimes, $\ell < \ell^*$ with $\alpha_1 = 0.64 \pm 0.02$ and $\ell > \ell^*$ with $\alpha_2 = 0.92 \pm 0.04$; these exponent values are similar to results of Liu et al. for the volatility. As an application, we use the scaling and memory properties of the return intervals to suggest a possibly useful method for estimating risk.

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1 Introduction

Interdisciplinary work has the potential to lead to results interesting for people from very different fields. In particular, collaborative work joining economists and physicists has resulted in a better understanding of economic fluctuations. Until relatively recently, theories of economic fluctuations invoked the label of "outliers" (bubbles and crashes) to describe fluctuations that do not agree with existing theory. However, recent research found evidence that the probability distribution of price fluctuations can be described by a power law [1–5]. Hence, there are no "outliers" since this law also holds for extremely large and unpredictable changes of magnitude sufficient to wreak havoc.

In economics, large and unpredictable fluctuations constitute risk for investments as well as the whole economy. For instance, in October 1929 the stock markets all over the world crashed, which initiated a worldwide economic crisis. However, significant risk could be inherent not only in worldwide market crashes, but also in less hazardous fluctuations if they are unexpected and investments are not well protected against them. Banks have to properly estimate the risk of their investments and make provisions in order to be able to withstand large fluctuations without going bankrupt.

In recent years, economic data bases have become available with a huge amount of data points, enabling physicists to analyze them as dynamic systems. The number of data points becames comparable to nano systems, but is still smaller than in bulk physical systems (say $\approx 10^8$, compared to Avogadro's number of $\approx 10^{23}$), but the "thermodynamic limit" is reached also for much smaller numbers so that methods from statistical physics can be applied to financial data. However, even in large data bases there is only a small amount of extremely large events so that they are still difficult to study. Hence, it is very important to find laws describing the entire data set, so that we can understand the extreme events (that matter!) by extensive analysis of small fluctuations (that do not matter).

Large events do not only occur in economics, but also appear in very different fields like climate or earthquakes. For instance, Gutenberg and Richter related huge earthquakes to everyday tremors in one single power law [6,7]. If one wants to prepare for an earthquake large enough to cause serious problems, it might be less important to exactly know how strong the next shock will be, but rather

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to know when a large shock will occur. A good approach is to study the time (return interval) between two successive shocks larger than a threshold above which a shock or fluctuation would damage a building or lead to bankruptcy of a bank, for instance. This way one can gather information of the temporal structure of the fluctuations.

Recently Bunde et al. [8–10] studied the statistics of return intervals and found that the long-term memory leads to a stretched exponential distribution and clustering of extreme events. They showed all these features in climate records and suggested that these phenomena should also occur in heartbeat records, internet traffic and stock volatility. Indeed, we find similar results in financial markets. We apply their scaling approach and find that the distribution of return intervals exhibits scaling properties for very different time scales and thresholds which seems to be universal for various financial time series. Furthermore, we analyze the short-term and long-term memory effects. We can apply these results in order to get a new method of risk estimation by predicting the future risk from the current return intervals.

2 Scaling and universality

Statistical physics deals with systems comprising a very large number of interacting subunits, for which predicting the exact behavior of the individual subunit would be impossible. Hence, one is limited to making statistical predictions regarding the collective behavior of the subunits. Recently, it has come to be appreciated that many such systems consisting of a large number of interacting subunits obey universal laws that are independent of the microscopic details. The finding, in physical systems, of universal properties that do not depend on the specific form of the interactions gives rise to the intriguing hypothesis that universal laws or results may also be present in economic and social systems [1,11]. An often-expressed concern regarding the application of physics methods to the social sciences is that physical laws are said to apply to systems with a very large number of subunits (of order 10^{20}), while social systems comprise a much smaller number of elements. However, the "thermodynamic limit" is reached in practice for rather small systems. For example, in early computer simulations of gases or liquids, reasonable results are already obtained for systems with 20–30 atoms.

2.1 Background

Suppose we have a small bar magnet made up of 10^{12} strongly-interacting subunits called "spins". We know it is a magnet because it is capable of picking up thumbtacks, the number of which is called the order parameter M. As we heat this system, M decreases and eventually, at a certain critical temperature T_c, it reaches zero. Since M approaches zero at T_c with infinite slope, the transition is remarkably sharp, hence M is not an analytic function. Such singular behavior is an example of a "critical phenomenon". Recently, the field of critical phenomena has been characterized by several important conceptual advances, two of which are scaling and universality.

2.2 Predictions of scaling

The scaling hypothesis has two categories of predictions, both of which have been remarkably well verified by a wealth of experimental data on diverse systems. The first category is a set of relations, called *scaling laws*, that serve to relate the various critical-point exponents characterizing the singular behavior of functions such as M.

The second category is a sort of *data collapse*, which is perhaps best explained in terms of our simple example of a uniaxial magnet. We may write the equation of state as a functional relationship of the form $M = M(H, \tau)$, where M is the order parameter, H is the magnetic field, and $\tau \equiv (T - T_c)/T_c$ is a dimensionless measure of the deviation of the temperature T from the critical temperature T_c. Since $M(H, \tau)$ is a function of two variables, it can be represented graphically and M vs. τ for a sequence of different values of H. The scaling hypothesis predicts that all the curves of this family can be "collapsed" onto a single curve provided one plots not M vs. τ but rather a *scaled* M (M divided by H to some power) vs. a *scaled* τ (τ divided by H to some different power).

The predictions of the scaling hypothesis are supported by a wide range of experimental work, and also by numerous calculations on model systems. Moreover, the general principles of scale invariance used here have proved useful in interpreting a number of other phenomena, ranging from elementary particle physics and galaxy structure to finance [1,12–14].

2.3 Universality

The second theme goes by the name "universality". It was found empirically that one could form an analog of the Mendeleev table if one partitions all critical systems into "universality classes". Consider, e.g., experimental MHT data on five diverse magnetic materials near their respective critical points. The fact that data for each material collapse onto a scaling function supports the scaling hypotheses, while the fact that the scaling function is the *same* (apart from two material-dependent scale factors) for all five diverse materials is truly remarkable. This apparent universality of critical behavior motivates the following question: *"which features of this microscopic interparticle force are important for determining critical-point exponents and scaling functions, and which are unimportant?"*

Two systems with the same values of critical point exponents and scaling functions are said to belong to the same universality class. Thus the fact that the exponents and scaling functions are the same for all five materials implies they all belong to the same universality class. Hence we can pick a tractable system to study and the results we obtain will hold for all other systems in the same universality class.

2.4 Scaling and universality in systems outside of physics

At one time, many imagined that the "scale-free" phenomena are relevant to only a fairly narrow slice of physical phenomena [15,16]. However, the range of systems that apparently display power law and scale-invariant correlations has increased dramatically in recent years, ranging from base pair correlations in noncoding DNA [17], lung inflation [18] and interbeat intervals of the human heart [19] to complex systems involving large numbers of interacting subunits that display "free will", such as city growth [20], university research budgets [21], and even bird populations [22].

3 Memory

Scaling and universality are important properties of a data set describing the global behavior of the probability distribution. This may, or may not, fully characterize a sequence of data points, depending on the time organization of the sequence. If it is *uncorrelated*, data points are independent of each other and totally determined by the probability distribution. On the other hand, if the points are *correlated*, it will also affect the order in the data set. This behavior is also called "memory", as the data points "remember" previous values.

Many studies showed that returns do not exhibit any linear correlations extending over more than a couple of minutes, but their absolute value, which is a measure for volatility, exhibits strong correlations. This leads to long periods of high volatility as well as other periods where the volatility is low. This effect is known as volatility clustering. We find similar effects also for return intervals, so that large (small) return intervals are more likely to be followed by large (small) return intervals.

4 Databases analyzed

Our results are based on the analysis of 5 different databases:

- (i) Trades and Quotes (TAQ) database for a 2-year period, from January 2, 2001 to December 31, 2002. It covers *all* securities traded in the three major US stock exchanges, namely, (a) the New York Stock Exchange (NYSE); (b) the American Stock Exchange (AMEX); and (c) the National Association of Securities Dealers Automated Quotation (Nasdaq). All 30 companies of the Dow Jones Industrial Average index (DJIA) are selected. The datasets of 160 000 points per DJIA stock with the sampling time of 1 min are analyzed.
- (ii) Standard and Poor's 500 index (S&P 500) for a 13-year period, from January 2, 1984 to December 31, 1996, with the sampling time of 10 min. The S&P 500 index, which consists of 500 companies, is a benchmark of the stock markets for the United States. Totally, 130 000 data points are studied.
- (iii) Daily stock prices from `http://finance.yahoo.com`. The website has the historical price for more than 6000 American stocks. For a typical stock like General Electric (GE), or International Business Machine (IBM), the website has the daily data for 44 years, from January 2, 1962 to now. The data size is about 11 000 data points each.
- (iv) Daily exchange rates of 35 other currencies to United States Dollar (USD) and federal funds rate from `http://www.federalreserve.gov`. The beginning and ending dates are different for those records. A typical rate, USD vs. Japanese Yen (JPY), starts from January 4, 1971 to now. The USD/JPY rate has around 9000 points for the 35-year period. Federal funds rate starts from July 1, 1954 to now, which has around 13 000 data points for the 52-year period.
- (v) Daily spot price of west Texas intermediate (WTI) crude oil from `http://www.eia.doe.gov` and daily gold price (London P.M.) from `http://www.onlygold.com`. Oil price starts from December 30, 1985 to now, totally around 5000 points. The range of gold price is from January 2, 1985 to now, the number of data points is about 5000.

5 Returns and volatilities

One basic measurement for changes in security prices, foreign exchange rates or other market quantities is their "return", giving the relative price change in a time interval. Returns show the speed and direction of the market movement. For example, if most returns are positive during a few years, the stock market is called to be a "bull market", while we call it a "bear market" if we observe mainly negative returns. The fluctuations in returns build the common behavior for financial markets, providing upside opportunities as well as downside risk to traders and investors. To characterize the volatile market, "volatility" is introduced as another fundamental concept. It describes the magnitude of the market fluctuations, irrespective of the direction.

5.1 Returns: scaling and universality

The nature of the distribution of price fluctuations in financial time series has been a topic of interest for over 100 years [23]. A reasonable *a priori* assumption, motivated by the central limit theorem, is that the returns are independent, identically Gaussian distributed (*i.i.d.*) random variables, which results in a Gaussian random walk in the logarithm of price.

Empirical studies [1–5,13,24–28] show that the distribution of returns has pronounced tails, in striking contrast to that of a Gaussian distribution. The cause of the power law tails is a subject of great current interest, involving the analysis of the price impact of orders and studies of limit order books [29–36]. In addition to being non-Gaussian, the process of returns shows another interesting property: "time scaling" — that is, the distributions of returns for various choices of Δt, ranging from one day up to even one year have similar functional forms [1]. These results together would suggest that the distribution of returns is consistent with a Lévy stable distribution [1,28,37–39], the rationale for which arises from the generalization of the central limit theorem to random variables which do not have a finite second moment. Empirical studies suggest, however, that the tails of the return distribution are inconsistent with the stable Paretian hypothesis [2–5,13,40–45]. In particular, alternative hypotheses for modeling the return distribution were proposed, which include a log-normal mixture of Gaussians [41], Student t-distributions [42–44], and exponentially-truncated Lévy distributions [13,46–48].

The basic quantity studied for individual companies with index i is the price $S_i(t)$. The time t runs over the working hours of the stock exchange — removing nights, weekends and holidays. For each company, we calculate the return

$$G_i \equiv G_i(t, \Delta t) \equiv \ln S_i(t + \Delta t) - \ln S_i(t). \quad (1)$$

For small changes in $S_i(t)$, the return $G_i(t, \Delta t)$ is approximately the forward relative change, $G_i(t, \Delta t) \approx [S_i(t + \Delta t) - S_i(t)]/S_i(t)$. Similar definitions are applied to other quantities like foreign exchange rates.

Previous empirical works studied the cumulative distributions — the probability of a return larger than or equal to a threshold — of returns G_i for some time intervals. For each stock, the asymptotic behavior of the functional form of the cumulative distribution is consistent with a power-law,

$$P\{G_i > x\} \sim x^{-\alpha_i}, \quad (2)$$

where α_i is the exponent characterizing the power-law decay. After normalizing the returns with their standard deviations in the 2-year period, which makes returns of different stocks comparable, Gopikrishnan et al. estimated the exponent α_i by a power-law regression. They obtained the average value $\alpha \simeq 3$ for the 1000 American stocks for both positive and negative tails of the distribution. Similar results are found in the analysis of daily returns of 30 German stocks composing the DAX index [2], foreign exchange rates [4], and daily CRSP returns [24].

5.2 Volatility and its correlations

Since volatility is supposed to describe the magnitude of the fluctuations, a direct definition of volatility is the absolute value of the return. When we study many datasets, different stocks or exchange rates have different sizes of returns or volatilities. To compare them, we define the volatility $g(t)$ as the absolute returns normalized by their standard deviation

$$g_i(t) \equiv \frac{|G_i(t)|}{(\langle G_i(t)^2\rangle - \langle |G_i(t)|\rangle^2)^{1/2}}, \quad (3)$$

where $\langle ... \rangle$ is the time average over the whole dataset.

In contrast to daily volatilities, the intraday data are known to show specific patterns [49–51], due to different trader behavior at different periods during the trading day. For example, the market is very active immediately after the opening [51], due to information arriving while the market is closed. The intraday pattern exhibits a pronounced peak at the opening hours, a minimum around the noon and a slight peak at the closing hours. This daily oscillation will cause some artificial correlations. One way to remove the intraday pattern is dividing the volatility by its average at the corresponding time of the day.

From the volatility definition, it is reasonable to conclude that the cumulative distribution of volatility also has a power-law tail, since both positive and negative tails of the cumulative distribution of returns are consistent with a power-law. Liu and collaborators [49] found that the cumulative distribution of volatility is consistent with power-law asymptotic behavior for the S&P 500 index and its 500 component stocks,

$$P(V_T^i > x) \sim x^{-\mu}, \quad (4)$$

where V_T^i is the average of $|G_i(t)|$ over a time window T, which is their definition of volatility. In the rest of this paper, we use equation (3) as the volatility definition. They find an exponent $\mu \simeq 3$, well outside the stable Lévy range $0 < \mu < 2$. For a systematic study of the PDF dependence on company size, see [52,53], and references therein.

Numerous studies analyzed the correlations of volatilities [24,49–51,54–62] which can be measured by the autocorrelation function (ACF). The volatility turns out to be long-term correlated, meaning that the ACF follows a power law

$$ACF \equiv \langle g_i(t) g_i(t + \tau)\rangle \sim \tau^{-\gamma}, \quad (5)$$

with exponent $\gamma \simeq 0.3$, while the autocorrelation for returns decays exponentially,

$$\langle G_i(t) G_i(t + \tau)\rangle \sim e^{-\tau/\tau_0}, \quad (6)$$

with the characteristic time $\tau_0 \simeq 4$ min for the S&P 500 index [49], for example.

More accurate results are obtained by detrended fluctuation analysis (DFA) [63–65]. This method is based on the idea that a correlated time series can be mapped to a self-similar process by integration. Therefore, measuring the self-similar feature can indirectly tell us information about the correlation properties. DFA permits the detection of long-range correlations embedded in a nonstationary time series, which is very common for records from financial markets. After removing trends, the DFA method computes the root-mean-square fluctuations $F(\ell)$ of time series within windows of ℓ points, and determines the correlation exponent α from the scaling function $F(\ell) \sim \ell^{\alpha}$.

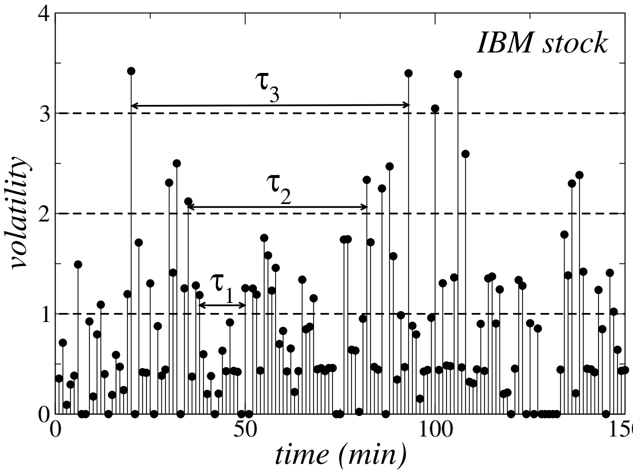

Fig. 1. Illustration of volatility return intervals. The volatility is normalized by its standard deviation. The solid circles are volatility values of the stock IBM on May 10, 2002. Return intervals τ_1, τ_2 and τ_3 for thresholds $q = 1$, 2 and 3 respectively are displayed. Adapted from [74].

The exponent α is related to the autocorrelation function exponent γ by

$$\alpha = 1 - \gamma/2. \tag{7}$$

If $\alpha > 0.5$, the time series has long-term correlations and exhibits persistent behavior, meaning that large (small) values are more likely to be followed by large (small) values. The value $\alpha = 0.5$ indicates that the signal is uncorrelated (white noise). While with $\alpha < 0.5$, the system has anti-correlation (or negative correlation). However, our results show some crossover in the volatility DFA curves. Split by $\ell^* = 390$, which represents 1 day, we find

$$\alpha_1 = 0.66 \pm 0.01 \tag{8a}$$

(group mean±standard deviation) for the short scale regime ($\ell < \ell^*$), and

$$\alpha_2 = 0.98 \pm 0.04 \tag{8b}$$

for the large scale regime ($\ell > \ell^*$) for 30 DJIA stocks and S&P 500 index, as shown in Figure 8. This result is consistent with earlier studies which also noted the crossover in DFA [49]. A similar crossover from short scales to large scales with similar values of α_1, α_2 has also been found for intertrade times [66].

6 Properties of volatility return intervals

The study of volatilities has been in the focus of econophysics for many years. In particular, understanding the variability in price movements may contribute to better risk estimation and portfolio management [67–73]. A new approach to this is analyzing the return interval, which is the time interval between volatilities above a certain threshold q [73–75]. Figure 1 illustrates the generation of return intervals. The time series of return intervals contains the temporal structure for those events. Also, it may

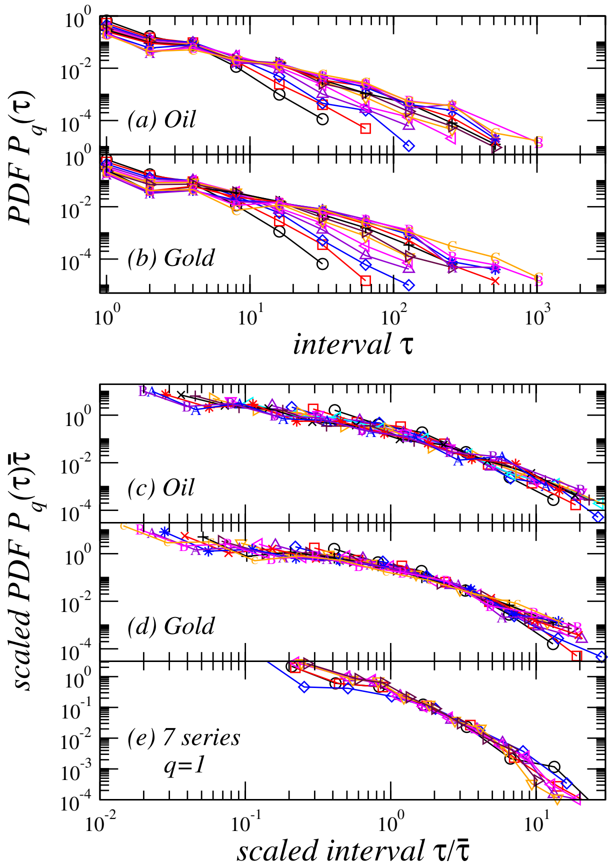

Fig. 2. (Color online) Distribution function $P_q(\tau)$ of the return intervals τ of the volatility records of (a) oil and (b) gold, for 13 threshold values q from 0.6 to 3.0 (○ 0.6, □ 0.8, ◇ 1.0, △ 1.2, ◁ 1.4, ▽ 1.6, ▷ 1.8, + 2.0, × 2.2, ∗ 2.4, A 2.6, B 2.8 and C 3.0). Figures (c) and (d) show the scaled plots where $P_q(\tau)\bar{\tau}$ is plotted versus $\tau/\bar{\tau}$. Figure (e) shows the scaled plots of 7 financial time series of commodity, interest rate, stock and currency for $q = 1.0$ (○ oil, □ gold, ◇ federal funds rate, △ IBM, ◁ S&P500, ▽ USD vs. JPY and ▷ UK Pound vs. Swiss Franc).

inherit the long-term properties of the volatility. Return intervals have also been studied in many other fields [76], like climate [77], seismic activities [78], solar flares [79], spikes in neurons [80] and turbulence in magnetic confined plasma [81]. It is calculated in similar ways but with different names, like waiting time, interocurrence time, interspike intervals, or laminar phases etc.

6.1 Scaling and universality of the distribution

In order to characterize the time series of return intervals $\{\tau\}$ between volatilities above a threshold q, we start with analyzing their probability density function (PDF), $P_q(\tau)$, similar to Yamasaki et al. [73]. Figure 2 shows $P_q(\tau)$ of return intervals for (a) Oil and (b) Gold for thresholds q between 0.6 and 3 standard deviations. The distributions seem to be very different for different thresholds, they get broader for larger thresholds since the intervals between volatilities above larger thresholds are longer than

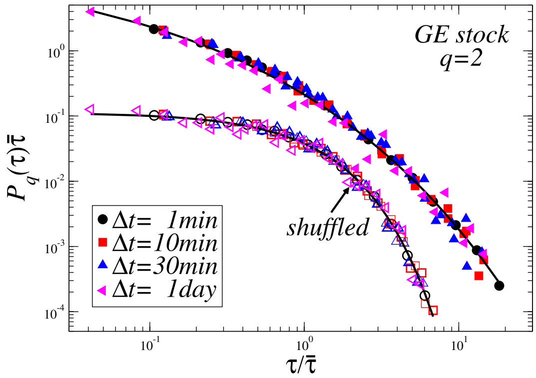

Fig. 3. (Color online) Scaled distributions of return intervals of original and shuffled data for different time resolutions Δt. The scaled PDF $P_q(\tau)\bar{\tau}$ of return intervals for GE stock is shown as a function of scaled time interval $\tau/\bar{\tau}$ for threshold $q = 2$. The sampling times are $\Delta t = 1, 10, 30$ min and 1 day. Filled symbols are for original data while open symbols correspond to shuffed data (down-shifted by factor 10). These symbols collapse onto two lines, one for original return intervals and one for shuffled data, which supports the scaling relation in equation (9). Also, the original return intervals suggest a stretched exponential scaling function, equation (11), since the line fitting the solid symbols is equation (11), with $a = 4.86$ and $\gamma = 0.30$. The stretched exponential is a result of the long-term correlations in the volatility records. The shuffled volatility records display no correlation, indicated by the good fit (solid line) to the Poisson distribution, equation (10).

for smaller values of q. Now the question is, are there any scaling relations between these PDFs?

If we scale the return intervals τ by the mean interval $\bar{\tau} \equiv \bar{\tau}(q)$ with threshold q, the PDFs of the scaled time series $\tau/\bar{\tau}$ approximately collapse onto a single curve as shown for Oil and Gold in Figures 2c and 2d. This suggests the existence of a scaling relation [73]

$$P_q(\tau) = \frac{1}{\bar{\tau}} f\left(\frac{\tau}{\bar{\tau}}\right). \qquad (9)$$

The scaling function $f(x)$ does not depend explicitly on q, but only through the mean return interval $\bar{\tau}$. Hence if $P_q(\tau)$ is known for one value of q, equation (9) can make predictions for other values of q — in particular for very large q (rare events), which are difficult to study due to the lack of data.

Figure 2e shows quite impressive the universality of the scaling relation equation (9) since it holds for a wide range of assets, namely commodities like oil and gold, the stock IBM, the stock market index S&P 500, and the foreign exchange rates USD vs. JPY and UK Pound vs. Swiss Franc for $0.6 \leq q \leq 3.0$.

For statistical analysis, the time resolution of the records is an important aspect since the system may exhibit diverse behavior in different time windows Δt. In Figure 3 we analyze four time scales for a typical stock, GE ($q = 2$), by combining the intraday and daily data. The figure shows that for $\Delta t = 1$, 10, 30 min and 1 day, the $P_q(\tau)\bar{\tau}$ curves collapse onto one curve, which shows the persistence of the scaling for a broad range of time scales. For larger thresholds like $q = 3$ or $q = 4$ the curves fluctuate more due to the limited data set. Thus there seems to be a universal structure for stocks not only in different companies, but also in each stock with various time resolutions. For a related study of persistence in different time scales of financial markets, see [82].

6.2 Form of scaling function

The shape of the PDF is of great interest for understanding complex systems because it contains abundant information about the underlying mechanics of the system. Surprisingly, empirical results show that the scaling function appears to be quite similar not only for currencies and commodities, but also for both daily and higher frequency stock and index data. The similarity indicates probably some "universal" structure in the time series for different thresholds and time resolutions. To understand this, we need to study the form of this scaling function $f(x)$.

Compared to the one-parameter Poisson distribution for uncorrelated records,

$$f(x) \sim e^{-x/x^*}, \qquad (10)$$

the scaling function of return intervals has a longer tail ("fat tail"), as shown in Figure 3. The fat tail could come from long-term correlations of the volatilities. We can test this if we destroy the correlations by randomizing the sequence of the series ("shuffling"). The return interval PDF becomes a Poisson distribution after shuffling the volatility records (Fig. 3). Hence, the long-term correlation is the reason for the fat tail.

Recently Bunde et al. simulated time series with long-term correlations and found that their return intervals exhibit a two-parameter stretched exponential scaling function [9],

$$f(x) \sim e^{-ax^\gamma}, \qquad (11)$$

where the exponent γ corresponds to the exponent of the autocorrelation function. Therefore, we test the stretched exponential for various datasets and compare it to a power-law function. By studying the daily datasets of seven stocks and seven currencies [73], Yamasaki et al. suggested that the scaling function is consistent with a power-law for $x \geq 1$, where the tail exponent is around 2 for both stock and currency data. The data are also consistent with the possibility that $f(x)$ is a stretched exponential for very large x-values. In this analysis, they study a range of thresholds q between 1 and 1.7.

Wang et al. analyzed the intraday data for 30 DJIA stocks and the S&P 500 index [74] and confirmed the similarity of the scaling function shape for these 31 datasets. Since the size of intraday data is more than 15 times larger than that of daily data for stocks and currencies (average 160 000 data points vs. around 10 000 points), they were able to analyze a much larger range of thresholds, $2 \leq q \leq 6$. They suggested that the data can be well fit by a stretched exponential form with $\gamma = 0.38 \pm 0.05$ and

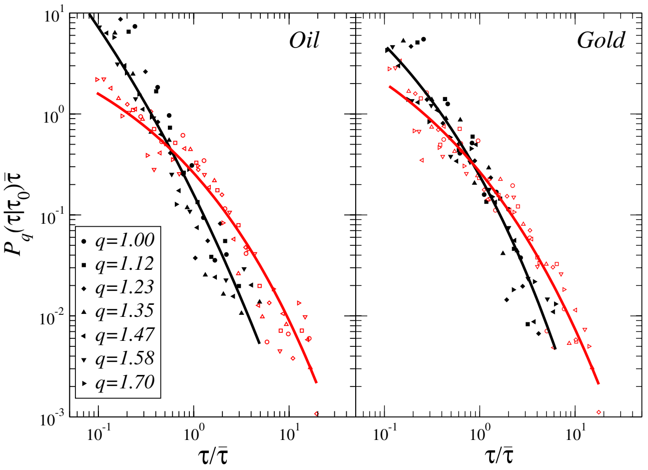

Fig. 4. (Color online) Conditional PDF $P_q(\tau|\tau_0)$ of the return intervals τ of the volatility records of the daily oil and gold time series, for τ_0 in the bottom octant Q_1 (full symbols) and in the top octant Q_8 (open symbols) versus scaled interval $\tau/\bar{\tau}$ for seven threshold values q from 1.00 to 1.70. The lines show nonliner fits to a stretched exponential $f(x) \sim e^{-ax^\gamma}$, equation (11), with exponents $\gamma = 0.27$ (Q_1) and 0.30 (Q_8) for oil and 0.25 (Q_1) and 0.25 (Q_8) for gold.

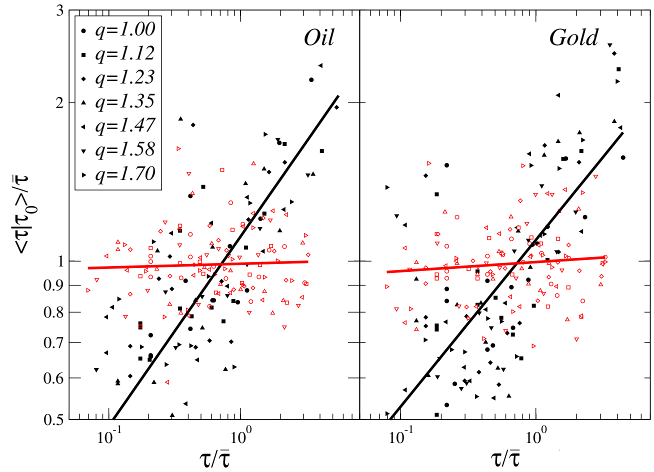

Fig. 5. (Color online) Mean conditional return interval $\langle\tau|\tau_0\rangle/\bar{\tau}$ versus $\tau_0/\bar{\tau}$ for volatility records for the daily oil and gold time series for seven threshold values q from 1.00 to 1.70. For records without memory, we expect $\langle\tau|\tau_0\rangle/\bar{\tau} \equiv 1$, as supported by the open symbols obtained for volatility shuffled records. The lines show a linear fit to the data.

$a = 3.9 \pm 0.5$. To examine the scaling for larger thresholds with good statistics, Wang et al. calculated the return intervals of each DJIA stock, and then aggregate all the data. They found that the scaling behavior could be extended even to the quite large threshold $q = 15$ (cf. Fig. 1, for which $q < 4$).

6.3 Short-term memory in the return interval sequence

To investigate the short-term memory in the return intervals of the records, we study the conditional PDF, $P_q(\tau|\tau_0)$, which is the probability of finding a return interval τ immediately after a return interval of size τ_0 [73]. In records without memory, $P_q(\tau|\tau_0)$ should be identical to $P_q(\tau)$ and independent of τ_0. Otherwise, it should depend on τ_0. Due to the poor statistics for a single τ_0, we study $P_q(\tau|\tau_0)$ for a bin (range) of τ_0. The entire database is partitioned into 8 equal-size subsets, Q_1, Q_2, ..., Q_8, with intervals in increasing length. Figure 4 shows the scaled $P_q(\tau|\tau_0)$ for oil and gold with $1.00 \leq q \leq 1.70$. For τ_0 in Q_1, the probability is larger for small τ, while for τ_0 in Q_8, the probability is higher for large τ. Thus, large (small) τ_0 tend to be followed by large (small) τ ("clustering"), which indicates memory in the return interval sequence. Thus, long-term correlations in the volatility records affect the PDF of intervals as well as the time organization of τ. Note also that for all thresholds $P_q(\tau|\tau_0)$ seems to collapse onto a single scaling function for each of the τ_0 subsets. Also, they can be well fit by a stretched exponential according to equation (11). The exponents γ are 0.27 for Q_1 and 0.30 for Q_8 for oil, and 0.25 (Q_1) and 0.25 (Q_8) for gold. These results are consistent with daily stocks and currencies data [73] and intraday stocks and index data [74].

Further, the short-term memory is also seen in the mean conditional return interval $\langle\tau|\tau_0\rangle$, which is the first moment of $P_q(\tau|\tau_0)$, immediately after a given τ_0 subset. $\langle\tau|\tau_0\rangle$ for oil and gold are shown in Figure 5 and GE stock in Figure 6. It shows again that large (small) τ tend to follow large (small) τ_0, similar to the clustering in the conditional PDF $P_q(\tau|\tau_0)$. Correspondingly, shuffled data (empty symbols) are almost constant, demonstrating that the value of τ is independent of the previous interval τ_0. The noise for these stock data (Fig. 6) is less than the noise for the data in Figure 5, presumably because the total number of data points in each dataset is 185 000 for GE, but only 5000 for oil and gold.

6.4 Clustering of return intervals

Clustering phenomena are displayed by $P_q(\tau|\tau_0)$ and $\langle\tau|\tau_0\rangle$, indicating time memory. However, both functions measure the intervals that immediately follow an interval τ_0. In order to investigate clustering in a more direct way, we analyze "clusters" of return intervals, which are composed by successive intervals with similar size [74,75]. To obtain good statistics we divide the sequence of return intervals into two bins, separated by the median of the entire database. We denote intervals that are above the median by sign "+", and the ones below the median by "–". Accordingly, n consecutive "+" or "–" intervals form a cluster.

The distribution of cluster size n may reveal more memory information in the sequence. Figure 7 shows the cumulative distribution of the size n for intraday data for GE stock. Both positive and negative clusters have quite long tails. For "+" clusters, the distribution still has good statistics to size $n = 18$, while the "–" clusters extend to $n = 25$. The memory effects persist for a quite long time

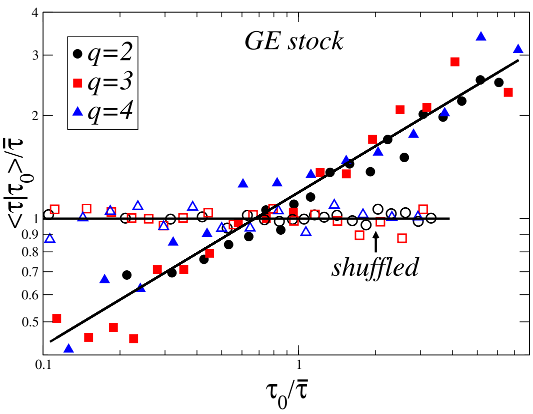

Fig. 6. (Color online) Mean conditional return interval $\langle\tau|\tau_0\rangle/\bar{\tau}$ vs. $\tau_0/\bar{\tau}$ for GE stock for three different thresholds $q = 2$, 3 and 4. The quantity $\langle\tau|\tau_0\rangle/\bar{\tau}$ (closed symbols) is compared with the same quantity for the shuffled records (open symbols). The distinct difference between the original data and the shuffled records implies there exist correlations (memory) in the original interval records. The noise for these stock data is less than the noise for the data in Figure 5, presumably because the total number of data in each dataset is 185 000 for GE (but only 5000 for oil and gold).

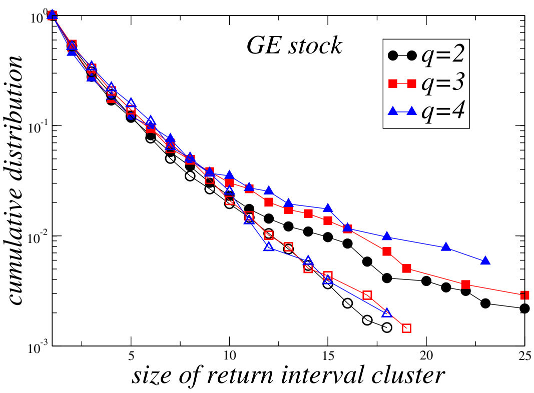

Fig. 7. (Color online) Cumulative distribution of size for return interval clusters, which consist of consecutive return intervals that are all above (open symbols) or below (closed symbols) the median of all the interval records. The GE stock is shown. Note that the two types of clusters have different tail behaviors.

(e.g., the average return interval for GE with threshold $q = 2$ is about 9.3 min, so there are still some clusters corresponding to even 200 min in time scale). We also note that the distribution of "+" clusters is very similar for different thresholds $q = 2, 3, 4$, while the "–" clusters show the same effect for $n \leq 10$. Similar clustering has been found also in earthquake and climate data [10,83].

6.5 Long-term memory in the return interval sequence

In the previous sections, we presented indications for short-term and medium-term memory in the return intervals sequence. Since in financial markets the volatility is known to have long-term correlations, the natural question is: can we detect long-term correlations in the return intervals? We apply the DFA method [63–65] to the return intervals, and the results are shown in Figure 8. Similar to volatilities, the DFA curves of return intervals of the S&P 500 index and 30 DJIA stocks can be split into two regimes $\ell < \ell^*$ and $\ell > \ell^*$ ($\ell^* \approx 93$ for return intervals, which corresponds to $\approx$1 day). We see that the corresponding values for α, α_1 and α_2 respectively, are distinctly different in the two regimes. Both α_1 and α_2 are larger than 0.5, suggesting long-term correlations in the return intervals time series, but they are not the same for different time scales.

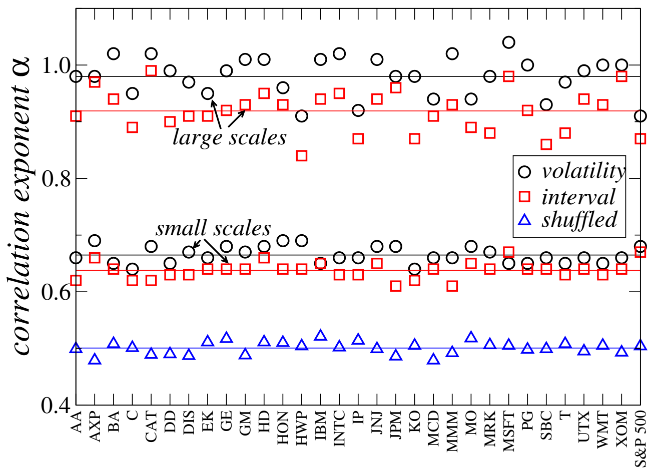

Fig. 8. (Color online) Correlation exponent α obtained from detrended fluctuation analysis (DFA) of volatility and return intervals. DFA curves (fluctuation $F(\ell)$ vs. ℓ) are fitted in two regimes, small scales ($\lesssim$1 day) and large scales ($\gtrsim$1 day) [49]. The correlation exponent α for 30 DJIA stocks and for the S&P 500 index (the stocks are identified by their tick symbols, e.g., AA denotes Alcoa Inc.) are shown. Note that most datasets have a smaller exponent for intervals than for volatilities, but their differences still are in the range of the error bars. Shuffled records (triangles) yield α values around 0.5 that indicate no correlation. Both small scales ($\alpha_1 = 0.66 \pm 0.01$ and $\alpha_1 = 0.64 \pm 0.02$, group average±standard deviation for volatilities and intervals, respectively, see Eqs. (8) and (12)) and large scales ($\alpha_2 = 0.98 \pm 0.04$ and $\alpha_2 = 0.92 \pm 0.04$ correspondingly) appear to show different correlations for different scales, since $\alpha_1 \neq \alpha_2$. Adapted from [74].

In the short scale regime ($\ell < \ell^*$), we find

$$\alpha_1 = 0.64 \pm 0.02, \tag{12a}$$

which is almost the same as for volatility [49] (the differences are within the error bars). In the large scale regime ($\ell > \ell^*$), we find

$$\alpha_2 = 0.92 \pm 0.04, \tag{12b}$$

and the differences between volatilities and return intervals are again in the range of the error bars. Here "error bars" refer to the error bars of each dataset which are on average $\approx$0.06, not the standard deviation of α_1 and α_2 as calculated from all 31 datasets we analyzed. Such behavior suggests a common origin for the strong persistence of correlations in both volatility and return interval records, and in fact the clustering in return intervals is related to the known effect of volatility clustering [84–86].

7 A method for risk estimation

After studying the statistics of return intervals, we now want to focus on a possible application. In particular, the scaling and memory properties of the return interval time series can be used for a new method of risk estimation [72]. The most common indicator of risk in the financial world is value-at-risk (VAR), which is defined by the risk at a "level of loss" Λ

$$\int_{-\infty}^{-\Lambda} p(G)dG = p^*, \tag{13}$$

where p^* is the probability of loss and $p(G)$ is the probability density function for returns $G(t)$.

In previous sections, we analyzed return intervals between events, where the absolute value of the return exceeds a threshold q. Now we focus on losses below $-q$ in order to estimate their risk. Since

$$\bar{\tau}_q \equiv \frac{1}{N_q}\sum_{i=1}^{N_q}\tau_q(i), \tag{14}$$

where

$$\sum_{i=1}^{N_q}\tau_q(i) \approx \text{total number of returns}, \tag{15}$$

and

$$N_q + 1 = \text{number of returns with } G < -q, \tag{16}$$

the average return interval $\bar{\tau}_q$ can be related to the VAR via equation (13) with $\Lambda = q$,

$${\tau_q}^{-1} = \int_{-\infty}^{-q} p(G)dG = \frac{\text{number of returns } G < -q}{\text{total number of returns}}. \tag{17}$$

This means that $\bar{\tau}_q^{-1}$ gives the loss probability for a risk level $-q$.

In the following we use the additional memory information contained in the sequence of $\{\tau_i\}$ to improve the estimation of the risk level of loss $-q$. First, we estimate the conditional mean return loss interval $\bar{\tau}_q(\tau_0)$ depending on the previous return interval τ_0. We also expect that, in analogy to equation (17)

$$\frac{1}{\bar{\tau}_q(\tau_0)} = \int_{-\infty}^{-q} p(G|\tau_0)dG. \tag{18}$$

Here $p(G|\tau_0)$ is the conditional probability that a return G will follow a return interval τ_0. The conditional mean return interval $\bar{\tau}_q(\tau_0)$ should be a straight line in a double logarithmic plot (see Fig. 6), so that

$$\log\left(\frac{\bar{\tau}_\mathrm{q}(\tau_0)}{\bar{\tau}_\mathrm{q}}\right) \propto \log\left(\frac{\tau_0}{\bar{\tau}_\mathrm{q}}\right). \tag{19}$$

Furthermore, $\bar{\tau}_q^{-1}$ scales like a certain power of q, $\bar{\tau}_q \sim q^{\psi}$ (e.g., $\psi = 3.3$ for IBM). Thus, the scaling properties of the return interval time series enable us to estimate $\bar{\tau}_q(\tau_0)$ also for large values of q. The memory in the return intervals as shown in Figure 6 gives us a more accurate estimation of the probability for a loss p^* in equation (13). Since the condition τ_0 changes every day, the risk level for the next day also changes according to the conditional mean $\bar{\tau}_q(\tau_0)$, which can be estimated with this method.

8 Discussion and conclusions

We analyzed the properties of the return intervals in financial time series. The scaling properties of the probability density function are very important because they help us to compensate the lack of data for large price fluctuations. Using the scaling relations, we can estimate the probability distribution of return intervals for large thresholds (extreme events) by analyzing the distribution derived from small thresholds.

We also found memory effects in the time series of return intervals. These correspond to the memory in the volatility time series, suggesting that they might be due to the same mechanism.

As an application, we use the scaling properties of the return intervals to get a method to estimate the risk of a certain loss. These results could lead to a better understanding and better modeling of risk in economics as well as in other research fields such as climate or earthquakes.

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